A Theoretical Analysis of PCC Stability

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Abstract

Performance-oriented Congestion Control (PCC) replaces hard-coded congestion-control rules with a sender-side optimization procedure: each sender experimentally changes its rate and keeps the direction that improves its measured utility. This note studies the game induced by the PCC utility function in a simple bottleneck-link model with multiple competing PCC senders. The main result is that, for a sufficiently sharp loss penalty, the induced game has a unique stable state, and that stable state is fair: all senders use the same rate. The note also analyzes the discrete PCC update dynamics and shows approximate convergence to a neighborhood of the stable state.

Context and scope

This technical report provides the theoretical analysis behind the stability behavior of PCC in a stylized multi-sender setting. The model intentionally abstracts away many implementation and network details in order to isolate one question: whether the PCC utility objective can induce a well-behaved equilibrium when several PCC senders share a single bottleneck. The report proves uniqueness and fairness of the stable state, then studies how the multiplicative increase/decrease dynamics approach that state. The proofs below are kept in their original technical form.

1 Model

Consider the following environment: n PCC senders $1, \dots, n$ send traffic across a bottleneck link of capacity $C > 0$. Each PCC sender i chooses a sending rate so as to optimize its utility function u_i, such that for every global configuration of sending rates $x = (x_1, \dots, x_n)$

$$u_i(x) = T_i(x) \cdot Sigmoid(L(x) - 0.05) - x_i \cdot L(x).$$

where $L(x) = \max\{0, 1 - \frac{C}{\Sigma_j x_j}\}$ is the loss rate, $T_i(x) = x_i(1 - L(x))$ is sender i's throughput, and $Sigmoid(y) \triangleq \frac{1}{1+e^{ay}}$ for some $\alpha > 0$, to be chosen later. Throughout the proofs we write a and α for the same steepness parameter. Without loss of generality, we set the link capacity to $C = 1$.

2 Unique Fair Stable State

We show here that this careful choice of utility function to embed into PCC induces a unique stable state of sending rates, that is, a state of sending rates $x^* = (x_1^*, \dots, x_n^*)$ such that no single sender can improve its performance by changing its sending rate. In game-theoretic terminology, this translates to the existence of a unique Nash equilibrium in this setting.

Theorem 1. *When $n \geq 3$ and $\alpha \geq \max\{2.2(n-1), 100\}$, there exists a unique stable state of sending rates $x_1, \dots, x_n$.*

Our experiments show that PCC indeed converges to this state. Our proof relies on analyzing three possible cases: (1) the sum of sending rates is less than the link capacity, that is, $\Sigma_j x_j < 1$; (2) $1 \leq \Sigma_j x_j \leq \frac{20}{19}$; and (3) $\Sigma_j x_j > \frac{20}{19}$. We first show that a stable state can exist only in case (2). We then show that there is a unique stable state in case (2).

We present the following simple claim, which pertains to the first of these three cases:

Claim 1. *Let $n \geq 3$ and $\alpha \geq \max(2.2(n-1), 100)$. Any configuration of sending rates $x = (x_1, \dots, x_n)$ such that $\Sigma_j x_j < 1$ is not a stable state.*

Proof. Consider a specific sender i, observe that when the sum of sending rates is strictly less than the link capacity, then according to the definition of i's utility function i can achieve better performance by increasing its transmission rate without exceeding the capacity. □

Claim 2. *Let $n \geq 3$ and $\alpha \geq \max(2.2(n-1), 100)$. Any configuration of sending rates $x = (x_1, \dots, x_n)$ such that $\Sigma_j x_j > \frac{20}{19}$ is not a stable state.*

Proof. The partial derivative of each sender i, when fixing the other senders' rates is:

$$\frac{\partial u_i(x)}{\partial x_i} = \frac{Sig(x)+1}{S(x)} - \frac{x_i(Sig(x)+1)}{S(x)^2} - \frac{\alpha x_i Y(x) Sig^2(x)}{S(x)^3} - 1 \tag{1}$$

Let $\Phi(x) \triangleq \sum_{i \in N} \frac{\partial u_i(x)}{\partial x_i}$. Summing the utilities over all senders we get,

$$\Phi(x) = \frac{(n-1)(Sig(x)+1)}{S(x)} - \frac{\alpha Y(x)}{S(x)^2} Sig^2(x) - n \tag{2}$$

Since $\Phi(x)$ depends only on $S(x)$, we abuse notation and denote eq. 2 by $\Phi(s)$ where $s = S(x)$.

Let $s = \Sigma_j x_j$. By (2), if $s > \frac{20}{19}$ then $\Phi(s) < 0$. Therefore, there must be at least one sender i with $\frac{\partial u_i(x)}{\partial x_i} < 0$ and hence x is not a stable state. □

Lemma 1. *Let $n \geq 3$ and $\alpha \geq \max(2.2(n-1), 100)$. Then, there exists a stable state x such that $\Sigma_j x_j \in (1, 20/19)$.*

Proof. By (2) we get

$$\Phi(1) = \frac{n-1}{1+e^{-0.05\alpha}} - \frac{\alpha e^{-0.05\alpha}}{(1+e^{-0.05\alpha})^2} - 1 \tag{3}$$

and as $Sig(x)|_{\Sigma_j x_j = \frac{20}{19}} = \frac{1}{2}$, $Y(x)|_{\Sigma_j x_j = \frac{20}{19}} = 1$ we have,

$$\begin{aligned}\Phi(\frac{20}{19}) &= \frac{19}{20} \cdot \frac{3}{2}(n-1) - \left(\frac{19}{20}\right)^2 \frac{\alpha}{4} - n \\ &= 0.425n - 0.225625\alpha - 1.425 \end{aligned} \tag{4}$$

In equation (3), we have that $\Phi(1) > 0$ for every $n \geq 3$ and $\alpha \geq 100$. In contrast, in equation (4), we have that $\Phi(\frac{20}{19}) < 0$ for every $n \geq 0$ and $\alpha \geq 2.2(n-1)$. Therefore, for every $n \geq 3$ and $\alpha \geq \max(2.2(n-1), 100)$ we have that $\Phi(1) > 0$ and $\Phi(\frac{20}{19}) < 0$. By the Intermediate Value Theorem, since $\Phi(s)$ is continuous in $s \in (1, \frac{20}{19}]$, there exists $\hat{s} \in (1, \frac{20}{19})$ such that $\Phi(\hat{s}) = 0$. Note that Claim 1 and Claim 2 imply that $\Phi(s) \neq 0$ if $s \notin [1, \frac{20}{19}]$. Therefore, $\Phi(s) = 0$ if and only if $s \in (1, \frac{20}{19}]$. Since any stable state x^* implies $\frac{\partial u_i(x^*)}{\partial x_i^*} = 0$, we get that $\Phi(S(x^*)) = 0$. Thus, it must be that $S(x^*) \in (1, 20/19)$. □

We now show that there is a unique stable state. This follows from the following auxiliary lemma.

Lemma 2. *Let $G(x)$ be an $n \times n$ matrix such that $G_{ij} = \frac{\partial^2 u_i(x)}{\partial x_i \partial x_j}$. If $S(x) \in [1, 20/19]$ and $a > 100$, then $G + G^T$ is negative definite.*

Proof. To prove that $G + G^T$ is strictly negative definite we show that both G and G^T are strictly negative definite. To show that G is strictly negative definite we define two matrices A and B such that $G = A + B$, and show that

A is negative semidefinite and B is strictly negative definite. Using the same arguments we show that $G^T = A^T + B^T$ is strictly negative definite.

First, we express G formally

$$G_{ij} \triangleq \frac{\partial^2 u_i(x)}{\partial x_i \partial x_j} = -\frac{sig(x)+1}{S(x)^2} + \frac{2x_i(sig(x)+1)}{S(x)^3} - \frac{\alpha Y(x) sig^2(x)}{S(x)^3} + \frac{4\alpha x_i Y(x) sig^2(x)}{S(x)^4} - \frac{x_i \alpha^2 Y(x) sig^2(x)}{S(x)^5} + \frac{2\alpha^2 x_i Y(x)^2 sig^3(x)}{S(x)^5} \tag{5}$$

Note that for every $j \neq i, k \neq i$, we have $G_{ij} = G_{ik}$ (all elements in each row are identical except for the diagonal element).

$$G_{ii} \triangleq \frac{\partial^2 u_i(x)}{\partial x_i^2} = -\frac{2(sig(x)+1)}{S(x)^2} + \frac{2x_i(sig(x)+1)}{S(x)^3} - \frac{2aY(x) sig^2(x)}{S(x)^3} + \frac{4\alpha x_i Y(x) sig^2(x)}{S(x)^4} - \frac{x_i \alpha^2 Y(x) sig^2(x)}{S(x)^5} + \frac{2\alpha^2 x_i Y(x)^2 sig^3(x)}{S(x)^5} \tag{6}$$

Let A be an $n \times n$ matrix such that

$$A_{ii} \triangleq -\frac{sig(x)}{S(x)^2} - \frac{\alpha Y(x)}{S(x)^3} sig^2(x) - \frac{1}{S(x)^2}$$
$$A_{ij} \triangleq 0 \quad \text{for } j \neq i$$

Let B be an $n \times n$ matrix such that all elements in each row i of B are identical and equal to G_{ij} (recall that G_{ij} for every $j \neq i, k \neq i$). Note that indeed $G = A + B$ and $G^T = A^T + B^T$ (since A is diagonal $A = A^T$). We will show that B is negative semidefinite. We first show that $B_{ij} < 0$ as for every $i \neq j$.

As $B_{ij} \triangleq G_{ij}$, simplifying (5) we get

$$
\begin{aligned}
B_{ij} &= \frac{2x_i - S(x)}{S(x)^3} sig(x) + \frac{4\alpha x_i Y(x) - \alpha Y(x) S(x)}{S(x)^4} sig^2(x) \\
&+ \frac{2\alpha^2 x_i Y(x)^2 - \alpha^2 x_i Y(x)(1+Y(x))}{S(x)^5} sig^3(x) - \frac{S(x) - 2x_i}{S(x)^3} \qquad (7) \\
&= \frac{x_i - r}{S(x)^3} sig(x) + \frac{\alpha Y(x)(3x_i - r)}{S(x)^4} sig^2(x) - \frac{\alpha^2 x_i Y(x)(1 - Y(x))}{S(x)^5} sig^3(x) + \frac{x_i - r}{S(x)^3} \qquad (8)
\end{aligned}
$$

where r denote $\sum_{j \neq i} x_j$.

There are two cases in (8):

1. If $x_i < \frac{S(x)}{4}$ then $B_{ij} < 0$ as all terms in (8) are negative.

2. Otherwise, we get

$$
\begin{aligned}
B_{ij} \leq & \frac{sig(x)}{S(x)^2} + \frac{\alpha Y(x)}{S(x)} sig^2(x) - \frac{\alpha^2 Y(x)(1 - Y(x))}{4S(x)^4} sig^3(x) + \frac{1}{S(x)^2} \\
< & \, 2 + \frac{\alpha Y(x)}{S(x)} sig^2(x) - \frac{\alpha^2 Y(x)(1 - Y(x))}{4S(x)^4} sig^3(x) < 0
\end{aligned}
$$

   where the last inequality is derived from the fact that by the specified constraints, the co-domain of $Y(x)$ is contained $[e^{-5}, 1]$, and $a \geq 100$.

Since all the elements in each row in B are identical and negative, the first eigenvalue $\lambda_1 = \sum_i \beta_i < 0$, and the other eigenvalues are zero. Therefore, B is negative semidefinite. In the same way, we can show that B^T is negative definite. In addition, A is a negative definite matrix as all its eigenvalues are negative. Note that $A = A^T$. Since G is a sum of a negative definite matrix A and negative semidefinite matrix B, G is negative definite. Similarly, since G^T is a sum of negative definite matrix A^T and negative semidefinite matrix B^T, G^T is negative definite. Hence, the matrix $G + G^T$ is negative definite as well. □

Lemma 3. *$u_i(x)$ is quasi-concave in x_i.*

Proof. To prove quasi-concavity, we show that $u_i(x)$ continuously increases in x_i up to some point $x_i = x_i^*$ and then $u_i(x)$ continuously decreases in x_i (assuming that $S(x_{-i}) \leq \frac{20}{19}$, as otherwise $u_i(x)$ decreases in x_i for all $x_i > 0$). We now divide the domain of $S(x)$ into four sub-domains $S(x) \leq 1$, $1 < S(x) < \frac{20}{19}$, $S(x) = \frac{20}{19}$, and $S(x) > \frac{20}{19}$. The high level idea is as follows: We first claim that

1. $\frac{\partial u_i(x)}{\partial x_i} > 0$ for $S(x) \leq 1$.

2. $\frac{\partial u_i(x)}{\partial x_i} < 0$ for $S(x) = \frac{20}{19}$.

By the Intermediate Value Theorem of $\frac{\partial u_i(x)}{\partial x_i}$, it must be that there is x^* for which $S(x^*) \in (1, \frac{20}{19})$ such that $\frac{\partial u_i(x^*)}{\partial x_i} = 0$. We will now show that for $1 < S(x) < \frac{20}{19}$, $\frac{\partial^2 u_i(x)}{\partial x_i^2} < 0$. This guarantees that there is a unique x_i such that $\frac{\partial u_i(x)}{\partial x_i} = 0$ (given a fixed sum of the others' sending rates). We will then show that $\frac{\partial u_i(x)}{\partial x_i} < 0$ for $S(x) > \frac{20}{19}$. Quasi-concavity will follow.

To simplify the argument for each subdomain, we substitute the terms in $\frac{\partial u_i(x)}{\partial x_i}$ with variables T_1, T_2, and T_3 as follows:

$T_1(x_i, x_{-i}) = \frac{Sig(x)+1}{S(x)}$

$T_2(x_i, x_{-i}) = 1 - \frac{x_i}{S(x)}$

$T_3(x_i, x_{-i}) = \frac{a x_i e^{a(0.95 - \frac{1}{S(x)})} Sig^2(S(x))}{S(x)^3}$

By (1), $\frac{\partial u_i(x)}{\partial x_i}$ can be expressed as $T_1 T_2 - T_3 - 1$. We now elaborate for each subdomain:

1. $\frac{\partial u_i(x)}{\partial x_i} > 0$ for $S(x) \leq 1$ (the case of no bottleneck. Then i benefits by increasing the sending rate - see Claim 1).

2. $\frac{\partial u_i(x)}{\partial x_i} < 0$ for $S(x) = \frac{20}{19}$ (it can be verified that $T_3 > T_1 \cdot T_2$ for $a > 100$).

3. $\frac{\partial^2 u_i(x)}{\partial x_i^2} < 0$ for $S(x) \in (1, \frac{20}{19})$. This follows as:

(a) T_1 decreases in x_i, as both $Sig(x)$ and $\frac{1}{S(x)}$ decreases in x_i, and T_2 decreases in x_i as $-\frac{x_i}{x_i+s}$ decreases in x_i for every constant $s > 0$. Therefore, $T_1 \cdot T_2$ decreases in x_i.

(b) T_3 increases in x_i as

$$\frac{\partial T_3(x)}{\partial x_i} = \frac{aY(x)Sig^2(x)}{S(x)^3}\left(1 - \frac{axY(x)Sig(x)}{S(x)^2} - \frac{3x}{S(x)} + \frac{ax}{S(x)^2}\right) \tag{9}$$

$$\geq \frac{aY(x)Sig^2(x)}{S(x)^3}\left(1 - \frac{ax}{4S(x)^2} - \frac{3x}{S(x)} + \frac{ax}{S(x)^2}\right) \tag{10}$$

$$\geq \frac{aY(x)Sig^2(x)}{S(x)^3}\left(1 + \frac{3x}{S(x)}\left[\frac{a}{4} \cdot \frac{19}{20} - 1\right]\right) \tag{11}$$

$$\geq 0 \tag{12}$$

where (9) follows since $Y(x) \cdot Sig(x) \leq \frac{1}{4}$ (this holds since $f(t) \triangleq \frac{t}{(1+t)^2} \leq \frac{1}{4}$ for all $t > 0$ and hence $Y(x)Sig(x) \equiv f(Y(x)) \leq \frac{1}{4}$). Inequality (10) follows as $S(x) \leq \frac{20}{19}$, and inequality 11 is due to $a > 100, S(x) > 0, Y(x) > 0$, and $Sig(x) > 0$.

4. $\frac{\partial u_i(x)}{\partial x_i} < 0$ for every $x > \frac{20}{19}$. This is true as the sigmoid function drops exponentially fast to zero as x_i increases. That is $\frac{\partial u_i(x)}{\partial x_i}|_{S(x)>\frac{20}{19}} \approxeq \frac{1}{S(x)}\left(1 - \frac{x_i}{S(x)}\right) - 1 < \frac{1}{S(x)} - 1 < 0$.

□

Proof of Theorem 1. Let $R_i = \{x_i \in \mathbb{R} | S(x) \in (1, \frac{20}{19}]\}$ be the possible sending rates of sender i. Since the constraints on each sender's rate are concave (that is $x_i > 0$ for all $i \in N$), $R = \prod_{i \in N} R_i$ is a convex set. Hence, as $G + G^T$ is a negative definite matrix (Lemma 2), according Rosen's theorem (Theorem 6 in [1]), the stable state in R is unique.

Theorem 2. *In the unique stable state $x^* = (x_1^*, \ldots, x_n^*)$ the sending rates of all senders are equal, i.e., $x_1^* = x_2^* = \ldots = x_n^*$.*

Proof. Suppose, for point of contradiction, that in the unique stable configuration there exist two senders, i and j, such that $x_i^* \neq x_j^*$. Observe, however, that as all senders have the same utility function, the state obtained from x^* by setting the sending rates of sender i to be x_j^* and the sending rate of sender j to be x_i^* must also be stable—a contradiction to the uniqueness of x^*. Thus, $x_1^* = x_2^* = \ldots = x_n^*$. □

3 Approximate Convergence to a Stable State

The utility of sender i can be expressed as

$$u_i(x_i, x_{-i}) = x_i \cdot \left(\frac{Sig(x_i, x_{-i}) + 1}{S(x_i, x_{-i})} - 1 \right)$$

where x_{-i} denotes the sending rates of the other senders,
i.e., $x_{-i} \triangleq (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$.

Claim 3. *Given a configuration of sending rates (state) $x = (x_1, \ldots, x_n)$ and two senders i, j, $\frac{\partial u_i(x)}{\partial x_i} > \frac{\partial u_j(x)}{\partial x_j}$ if $x_j > x_i$.*

Proof. For any sender $p \in N$, $u_p(x)$ depends only on x_p and $S(x)$. Since $S(x)$ is the same for all senders, in order to see how the utility changes for a given state, we explore how the utility increases given a fixed $S(x)$ (and denote $S(x) \triangleq S$). Therefore,

$$\frac{\partial u_p(x)}{\partial x_p} = \frac{Sig(S) + 1}{S} - 1 - x_p \left(\frac{1}{S} + \frac{a e^{a(0.95 - \frac{1}{S})} Sig^2(S)}{S^3} \right) \tag{13}$$

Since, (13) is a (linear) decreasing function in x_p, the claim follows. □

3.1 PCC dynamics

Let x_i^t denote the sending rate (SR) of sender i at time t. The PCC dynamic is as follows: at each time step t, j updates his sending rate according to

$$x_j^{t+1} = \begin{cases} x_j^t(1+r) & \text{if } u_j(x_j^t(1+r), x_{-j}) > u_j(x_j^t(1-r), x_{-j}) \\ x_j^t(1-r) & \text{if } u_j(x_j^t(1-r), x_{-j}) > u_j(x_j^t(1+r), x_{-j}) \end{cases}$$

where $r > 0$ is small (about 0.01). We denote the increase step of x_j by $x_j \uparrow$ and the decrease step by $x_j \downarrow$.

We define an "indifferent state" $v^* = (\hat{v}, ..., \hat{v})$ to be a configuration of sending rates in which all PCC senders are indifferent between increasing or decreasing their sending rate. We will show that such a state indeed exists. Let $\bar{x}$ be the average sending rate, i.e., $\bar{x} \triangleq \frac{S(x)}{n}$.

Lemma 4. *(monotonicity) Let x be a state, and let $i, j \in n$ be a pair of senders such that $x_j > x_i$.*

1. *If $x_j \uparrow$ then $x_i \uparrow$.*

2. *If $x_i \downarrow$ then $x_j \downarrow$.*

Proof. Note that

$$\begin{aligned} u_i(x_i(1+r), x_{-i}) - u_i(x_i, x_{-i}) = & \int_{x_i}^{x_i(1+r)} \tfrac{\partial u_i(l, x_{-i})}{\partial l} \, dl \\ u_i(x_i(1-r), x_{-i}) - u_i(x_i, x_{-i}) = & \int_{x_i}^{x_i(1-r)} \tfrac{\partial u_i(l, x_{-i}))}{\partial l} \, dl \end{aligned}$$

Hence, $x_i^{t+1} = x_i^t(1+r)$, if and only if

$$\int_{x_i(1-r)}^{x_i(1+r)} \frac{\partial u_i(l, x_{-i})}{\partial l} \, dl > 0 \tag{14}$$

Suppose that $x_j \uparrow$. We now check two complementary cases:

1. If $\frac{\partial u_i(x_i(1+r),x_{-i})}{\partial l} > 0$, this implies that $\frac{\partial u_i(l,x_{-i})}{\partial l} > 0$ for all $l \in (x_i(1-r), x_i(1+r))$. Hence, condition (14) holds. Note that in this case the condition holds independently of j's action.

2. Otherwise (that is $\frac{\partial u_i(x_i(1+r),x_{-i})}{\partial l} < 0$), we show that for every $x_j > x_i$, sender i benefits more by increasing than sender j, that is,

$$\int_{x_j(1-r)}^{x_j(1+r)} \frac{\partial u_j(l, x_{-j})}{\partial l}\, dl - \int_{x_i(1-r)}^{x_i(1+r)} \frac{\partial u_i(l, x_{-i})}{\partial l}\, dl < 0 \tag{15}$$

Specifically:

$$\begin{align} & \int_{x_j(1-r)}^{x_j(1+r)} \frac{\partial u_j(l, x_{-j})}{\partial l}\, dl - \int_{x_i(1-r)}^{x_i(1+r)} \frac{\partial u_i(l, x_{-i})}{\partial l}\, dl \tag{16} \\ < & \int_{x_i - rx_j}^{x_i + rx_j} \frac{\partial u_i(l, x_{-i})}{\partial l}\, dl - \int_{x_i - rx_i}^{x_i + rx_i} \frac{\partial u_i(l, x_{-i})}{\partial l}\, dl \tag{17} \\ = & \int_{x_i + rx_i}^{x_i + rx_j} \frac{\partial u_i(l, x_{-i})}{\partial t}\, dl - \int_{x_i - rx_i}^{x_i - rx_j} \frac{\partial u_i(l, x_{-i})}{\partial l}\, dl \tag{18} \\ & < 0 \tag{19} \end{align}$$

Inequality (17) follows from Claim 3. Inequality 19 is derived as follows: The left term satisfies $\int_{x_i+rx_i}^{x_i+rx_j} \frac{\partial u_i(l,x_{-i})}{\partial t}\, dl < 0$. To see this, observe that $x_i(1+r)$ is on the "decreasing part" of the function and therefore $x_j(1+r)$ is on the "decreasing part". Hence $\frac{\partial u_i(l,x_{-i})}{\partial l} < 0$ for all $l \in (x_i + rx_i, x_i + rx_j)$. The right term satisfies $\int_{x_i-rx_i}^{x_i-rx_j} \frac{\partial u_i(l,x_{-i})}{\partial l}\, dl > 0$. This is derived from the assumption that $x_j \uparrow$, which implies that $\frac{\partial u_i(l,x_{-i})}{\partial l} > 0$ for all $l \in (x_i - rx_j, x_i - rx_i)$.

□

From the monotonicity property we can derive the following possible dynamics:

Corollary 1. *1. All senders increase their SR (called total increase).*

2. *All senders decrease their SR (called total decrease)*

3. *There is a "cross-over" rate p such that each sender i with $x_i > p$ decrease its SR, and each sender i with $x_i < p$ increase its SR.*

Another important property relates to the average of the senders SR.

Lemma 5. *The PCC dynamics are such that*

1. *If $\bar{x} \geq \hat{v}$ and $x_i > \hat{v}$ then $x_i \downarrow$.*

2. *If $\bar{x} \leq \hat{v}$ and $x_i < \hat{v}$ then $x_i \uparrow$.*

Proof. We only prove the first statement as the second statement relies a symmetric argument. Let x be a state such that $\bar{x} \geq \hat{v}$ and let x' be a state such that $\bar{x}' = \hat{v}$. Suppose that $x_j = x'_j = \hat{v}$ for some sender j. Since $\bar{x}' \leq \bar{x}$, the link at x is more congested than at x' and so the increase in utility from $x_j \uparrow$ is lower then the increase in utility from $x'_j \uparrow$. Since x'_j is indifferent, i.e., $u_j(x'_j \uparrow, x_{-j}) = u_j(x'_j \downarrow, x_{-j})$ (as $x_j = \hat{v}$ and $\bar{x}' = \hat{v}$), it must be that j benefits by $x_j \downarrow$. If $x_i > \hat{v}$ then $x_i > x_j$, and hence by Lemma 4, i benefits by $x_i \downarrow$. $\square$

We now classify the possible dynamics as follows:

1. Increasing cycle. A sequence of states $R = (x_1, .., x_T)$, where $T \geq 2$, is called increasing cycle if all senders in all states in R increase their SR, and at least one sender in x_{T+1} decreases its SR.

2. Decreasing cycle. A sequence of states $R = (x_1, .., x_T)$, where $T \geq 2$ (except for the special case of backing up, where $T \geq 3$), is called a decreasing cycle if all senders at all states in R decrease their SR, and at least one sender in x_{T+1} increases its SR.

3. ZigZag: all senders increase their SR and then decrease their SR alternately. We call the series of the zigzag actions a cycle. A series of states

$x^T, x^{T+1}, ..., x^{T+a}$ is called ZigZag cycle if $x^{t+1} = x^t(1+r)$ (namely all senders increase their SR) for $t \in (1, 3, 5, ..., a-1)$ and $x^{t+1} = x^t(1-r)$ (namely all senders decrease their SR) for $t \in (2, 4, 6, ..., a)$. The ZigZag cycle is the maximal series of 'zigzags' in the sense that X^{T-1} increases its SR and x^{T+a+1} decreases it SR. A state x^t that obtained by all senders increase their SR is called *high peak* and a state x^t that obtained by all senders decrease their SR is called *low peak*.

4. Crossover: This is a one step dynamics. Let S_1 and S_2 be a partition of N such that for each $i \in S_1$ and $j \in S_2$, $x_i < x_j$. Under crossover we have $x_{i\in S_1} \uparrow$ and $x_{j\in S_2} \downarrow$.

5. Backing up: Two consecutive steps of total increase that begins when $\bar{x} \in (1-r)\hat{v}, \frac{1}{1+r}\hat{v}))$.

6. ZigZag - Backing up - ZigZag (ZBZ): A ZigZag cycle that is followed by two consecutive states in which all senders increase their SR after a ZigZag cycle (backing up) and then another ZigZag.

Let $\hat{C} \triangleq ((1-r)\hat{v}, (1+r)\hat{v})$ be a domain of sending rates.

Lemma 6. *In a ZigZag cycle R, for all $t \in R$, $\bar{x}^t \in \hat{C}$.*

Proof. Suppose, by contradiction, that $\bar{x} \notin \hat{C}$. W.l.o.g, suppose that for t, $\bar{x}^t = a$ where $a > (1+r)\hat{v}$. There are two cases:

1. Suppose that x^t is a high peak. Thus at $t-1$ there is total increase. Hence, $\bar{x}^{t-1} = \frac{a}{1+r} > \hat{v}$, and w.l.o.g., the maximal sender $x^{t-1}_{max} > \bar{x}$. But then $x^{t-1}_{max} > \hat{v}$ and $\bar{x}^{t-1} > \hat{v}$. But by the property of Lemma 5 $x^{t-1}_{max} \uparrow$ contradiction.

2. Otherwise, suppose that x^t is a low peak (i.e., total increase at time t). Again, w.l.o.g., the maximal sender $x^t_{max} > \bar{x}$, and then $x^{t+1}_{max} > \hat{v}$. Since

$\bar{x}^t > \hat{v}$ and $x_{max}^{t+1} > \hat{v}$, by the property of Lemma 5, $x_{max}^{t+1} \downarrow$ - contradiction.

□

The backing up dynamics can occur after a ZigZag cycle (as the sending rates of all senders decrease after each pair of total increase-total decrease cycle) until $\bar{x} \in (1-r)\hat{v}, \frac{1}{1+r}\hat{v})$ which then ZigZag cannot occur (otherwise it contradicts Lemma 6). If after a backing up there is another ZigZag cycle then it is ZBZ. Endless ZBZ (namely, zigzag cycle that ends in backing up and then return to another zigzag cycle and so on infinitely) is an unwanted behavior (as it contradicts convergence). But we claim that this 'bad' dynamics cannot occur in general.

Lemma 7. *There is no endless ZBZ (for almost all possible values of r).*

Proof. We claim this for a general r (however it might happen for an exact value of r). Each zigzag cycle starts in a slightly different $\bar{x}$ after backing up. Hence at some zigzag cycle, one of the high peaks of $\bar{x}$ in the ZigZag cycle will be sufficiently close to $\hat{v}$ and so there will be a crossover (by Lemma 9) (this, for instance, can be enforced by choosing an irrational value for r; then in the ZBZ cycle, $\bar{x}$ forms a dense set in $\hat{C}$). □

Lemma 8. *For any initial state there is at most increasing or decreasing cycle (but not both) which begins at the initial state.*

Proof. Assume there is a decreasing cycle that begins at time t, after the following dynamics:

1. **Increasing cycle**. Then at $t-1$ there is a total increase and at $t+1$ there is a total decrease. Therefore, for every sender i, $x_i^{t+1} = (1+r)(1-r)x_i^{t-1} = (1-r^2)x_i^{t-1}$, and then the average satisfies $\bar{x}^{t+1} = (1-r^2)\bar{x}^{t-1}$ as well. Because $x_i^{t+1} < x_i^{t-1}$ and $\bar{x}^{t+1} < \bar{x}^{t-1}$, as each sender $x_i^{t-1} \uparrow$, x_i^{t+1} would

benefit more than x_i^{t-1} by increasing SR. Thus, as $x_i^{t-1} \uparrow$ it must be that $x_i^{t+1} \uparrow$ - contradiction to assumption.

2. **ZigZag** (or ZBZ). Followed by the exact same argument as the increasing cycle case, it is impossible that a decreasing cycle will occur after ZigZag.

3. **Crossover**. Let x_{min} be the sender with minimal SR at time $t-1$. Crossover at time $t-1$ implies that $x_{min}^{t-1} \uparrow$. Therefore, $x_{min}^{t+1} = (1+r)(1-r)x_{min}^{t-1} = (1-r^2)x_{min}^{t-1}$. By crossover, there are senders at time $t-1$ that decrease SR and there are sender that increase SR (in particular x_{min}), hence using similar argument as in the increasing cycle case, we get $\bar{x}^{t+1} < (1-r^2)\bar{x}^{t-1}$ (but now its a strict inequality instead of equality) and $x_{min}^{t+1} \uparrow$ - contradiction to assumption.

The proof that an increasing cycle cannot occur after Decreasing cycle, ZigZag, and crossover is followed by the symmetric argument (unless a condition for backing up is met and then comparing x^{t-1} and x^{t+2}) □

We call the increasing or decreasing cycle a *starting phase.*

Therefore, by Lemma 8 after the initial phase of possible Increasing or Decreasing cycle, the only possible dynamics are ZigZag, Crossover and Backing-up (less probable).

3.1.1 Crossover

The following lemma tells us when crossover occurs.

Lemma 9. *If $\bar{x}$ is sufficiently close to $\hat{v}$ (including $\hat{v}$ itself) then crossover occurs.*

Proof. Let p be a crossover pivot such that if $x_j > p$ then $x_j \downarrow$ and if $x_j < p$ then $x_j \uparrow$. Because the utility function is quasi concave and $(\hat{v}, ..., \hat{v})$ is a stable state, if $\bar{x} = \hat{v}$ then each sender with SR lower than $\hat{v}$ will benefit by increasing the SR, and each sender with SR greater than $\hat{v}$ will benefit by decreasing the SR. Hence, $p = \hat{v}$. In addition, note that increasing $\bar{x}$ continuously results in a continuous decrease in p (as the link becomes more congested, it becomes less attractive to increase the sending rate for all senders). Since any continuous change in $\bar{x}$ causes a continuous change in p, for infinitesimal change in $\hat{x}$, $p > 0$. However, note that if $\bar{x}$ is "far" from $\hat{v}$ then $p < 0$ and there is no crossover. □

The following claim (which follows immediately by the definition of crossover) says that if the two sending rate are not too close then after crossover they become closer. Formally,

Claim 4. *Let t be a time step in which a crossover occurs, where $x_i^t \downarrow$ and $x_j^t \uparrow$. If $x_i^t(1-r) > x_j^t(1+r)$, then the distance between the two senders decreases (i.e., $|x_i^{t+1} - x_j^{t+1}| < |x_i^t - x_j^t|$).*

We now check what if the distance of two sending rate by crossover can increase when x_i is "close" to x_j.

Claim 5. *If two senders i, j such that $x_i^t > x_j^t$ undergo crossover, so that $x_i^t \downarrow$ and $x_j^t \uparrow$, then $x_j^{t+1} < x_i^{t+1} \frac{1+r}{1-r}$.*

Proof. After crossover, $x_i^{t+1} = x_i^t(1-r)$ and $x_j^{t+1} = x_j^t(1+r)$. Since $x_j^t < x_i^t$, we have

$$x_j^{t+1} = x_j^t(1+r) < x_i^t(1+r) = x_i^{t+1} \frac{1+r}{1-r}.$$

For small r, the factor $\frac{1+r}{1-r}$ is approximately $(1+r)^2$. □

Let x_{max} be the maximal SR, and x_{min} be the minimal SR.

Lemma 10. *There is a finite number of crossovers such that after that $x_{max}^{t+1} < x_{min}^{t+1} \frac{1+r}{1-r}$, and $x_{min}^{t+1} > x_{max}^{t+1} \frac{1-r}{1+r}$.*

Proof. By Claim 4, for all senders that are far apart, their distance is reduced. By Claim 5, any pair of senders that is close remains close up to the factor $\frac{1+r}{1-r}$. Since ZigZag does not increase the distance between senders, after a finite number of steps we get $x^{t+1}_{max} < x^{t+1}_{min}\frac{1+r}{1-r}$. The lower bound follows by the symmetric argument. □

If state x satisfies $x^{t+1}_{max} < x^{t+1}_{min}\frac{1+r}{1-r}$, and $x^{t+1}_{min} > x^{t+1}_{max}\frac{1-r}{1+r}$ then we call this convergent state.

Lemma 11. *If all senders are at convergent state, then* $\bar{x} \in \hat{C}$

Proof. Since by Lemma 6, during ZigZag we always have $\bar{x} \in \hat{C}$, and ZBZ is possible only when $\bar{x} \in ((1-r)\hat{v}, \frac{1}{1+r}\hat{v})$ and hence after backing up $\bar{x} < (1+r)\hat{v} \in \hat{C}$, we only need to show that after a crossover it holds that $\bar{x} \in \hat{C}$. Suppose w.l.o.g. that after a crossover $\bar{x}^{t+1} > \hat{C}$. This implies that $\bar{x}^t > \hat{v}$, as otherwise it wouldn't increase to this high SR at time $t+1$ (we assume $\bar{x}^t \in \hat{C}$ as, w.l.o.g, it occurs after a ZigZag cycle). Since the average increases at time t, there must be a sender i that increases it SR, and by lemma 5, it must be that $x_i < \hat{v}$. Since all senders are at convergent state it implies that $x^{t+1}_i > \bar{x}$ (as all senders that are above $\hat{v}$ decrease their SR). Therefore, i increases its rate by more than $(1+r)$ in one step (as $x^t_i < \hat{v}$ and $x^t_i > \hat{v}(1+r)$), which is clearly impossible. □

Theorem 3. *There is a time t' such that for all $t > t'$,*
$x^t_i \in \left(\frac{(1-r)^2}{1+r}\hat{v}, \frac{(1+r)^2}{1-r}\hat{v}\right)$ *for each $i \in N$.*

Proof. By Lemma 8, after the initial phase the only possible dynamics are ZigZag (or ZBZ) and crossover. By Lemma 7 there is no endless ZBZ. This implies that there is infinite number of crossovers. Therefore, by Lemma 10 after a finite number of steps all senders are in convergent state. Since by Lemma 11 $\bar{x} \in \hat{C}$, the theorem follows □

References

[1] J.B Rosen. Existence and uniqueness of equilibrium point for concave n-person games. Econometrica, 1965.